URYSOHN'S LEMMA

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1. Definitions

Definition. *A topological space X is **locally compact** provided that every point $x \in X$ has some open neighborhood $V_x \subset X$ with compact closure $\overline{V}_x$.*

Definition. *A topological space X is **Hausdorff** provided that, given two distinct points x and y in X, there are disjoint open subsets A and B of X such that $x \in A$ and $y \in B$.*

Fact. *In a Hausdorff space X, given any disjoint compact subsets A and B of X, there exist disjoint open subsets C and D of X such that $A \subset C$ and $B \subset D$.*

2. Urysohn's Little Lemma

Paul Garrett's proof of the Urysohn's Lemma relies heavily on one claim, which I'll demonstrate first.

Lemma. *In a locally compact Hausdorff topological space X, given a compact subset K contained in an open set U, there exists some open set V with compact closure $\overline{V}$ such that*

$$K \subset V \subset \overline{V} \subset U.$$

Proof. First, by local compactness, we can choose for each $x \in K$ some open neighborhood V_x with compact closure. The union of all these V_x covers K; in fact, since K is compact, we can reduce this cover to a finite collection $\bigcup_{i=1}^{n} V_{x_i}$.

We'll call this finite union W, noting that since each $\overline{V}_{x_i}$ is compact, since a finite union of compact sets is compact, and since closure commutes with finite unions, W has *compact* closure. Now we have

$$K \subset W \subset \overline{W}, \qquad W = \bigcap_{i=1}^{n} V_{x_i}, \quad \overline{W} = \bigcap_{i=1}^{n} \overline{V}_{x_i}.$$

In particular, it will be useful later that $K \subset W$.

Similarly, for any $x \in U^c$ (recall $x \in U^c \Rightarrow x \notin K$) we can find an open neighborhood U_x of x which is disjoint from some corresponding open W_x containing K. (This is justified by the fact above.)

Now we have a cover of U^c

$$U^c \subset \bigcup_{x \in U^c} U_x$$

and since each U_x manages to completely miss the corresponding $W_x \supset K$, we have

$$(\bigcup_{x \in U^c} U_x) \cap (\bigcap_{x \in U^c} W_x) = \phi.$$

Note that since each open W_x is completely disjoint from our useful open neighborhood U_x of $x \in U^c$, and our space is Hausdorff, the closure $\overline{W}_x$ misses U_x as well, and thus $\bigcap \overline{W}_x$ manages to miss all $x \in U^c$. So in particular,

$$U^c \cap (\bigcap_{x \in U^c} \overline{W}_x) = \phi.$$

By rearranging this intersection and throwing in $\overline{W}$ for free, we get

$$\bigcap_{x \in U^c} (U^c \cap \overline{W} \cap \overline{W}_x) = \phi.$$

Now, since $\overline{W}$ is compact, all of the terms of the intersection above are compact, and so a finite intersection suffices for an empty intersection:

$$(\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_m}) \cap U^c = \phi.$$

We can conclude that

$$\overline{W} \cap \overline{W}_{x_1} \cap \ldots \cap \overline{W}_{x_m} \subset U.$$

Now let $V = W \cap W_{x_1} \cap \ldots \cap W_{x_m}$. Since $K \subset W$ and $K \subset W_{x_i}$ for all i, we have $K \subset V$. Thus we have, as desired,

$$K \subset V \subset \overline{V} \subset U.$$

□

3. Semi-continuity and Continuity

Definition. *A (real-valued) function $f : X \to \mathbb{R}$ is said to be* ***lower semi-continuous*** *if the set $\{x : f(x) > B\}$ is open for all $B \in \mathbb{R}$. A (real-valued) function $f : X \to \mathbb{R}$ is said to be* ***upper semi-continuous*** *if $\{x : f(x) < B\}$ is open for all $B \in \mathbb{R}$.*

Proposition. *Any (pointwise) supremum of lower semi-continuous functions is itself lower semi-continuous. A pointwise infimum of upper semi-continuous functions is upper semi-continuous.*

Proof. Let G be a set of real-valued lower semi-continuous functions g, and let f be the (pointwise) supremum of all g. Fix some arbitrary $B \in \mathbb{R}$, and let

$$Q = \{x \in X : f(x) > B\}$$

and

$$M = \bigcup_{g \in G} \{x \in X : g(x) > B\}.$$

Since M is an arbitrary union of open sets, M is itself open. I'll show that $M = Q$.

Consider any point $x \in X$. Then $x \in M$ if and only if our chosen B is *not* an upper bound on $\{g(x) : g \in G\}$. This is equivalent to the statement that $f(x) > B$, since being less than the least upper bound is precisely what disqualifies B from being an upper bound. Concluding that $f(x) > B$, though, is exactly saying that $x \in Q$.

So $M = Q$. Thus $\{x : f(x) > B\}$ is open for all $B \in \mathbb{R}$. It follows that f is lower semi-continuous.

The second statement follows from the first. □

Remark. *If a function is both lower semi-continuous and upper semi-continuous, then it is also continuous. Conversely, if a function is continuous, then it is both lower semi-continuous and upper semi-continuous.*

Proposition. *The characteristic function of an open set is lower semi-continuous, and the characteristic function of a closed set is upper semi-continuous.*

Proof. Let A be an open set and B a closed set, both in a topological space X. Then for $\alpha \in \mathbb{R}$,

$$\{x : \chi_A(x) > \alpha\} = \begin{Bmatrix} \phi & \alpha \geq 1 \\ A & 0 \leq \alpha < 1 \\ X & \alpha < 0. \end{Bmatrix}$$

Since X, A, and ϕ are all open, χ_A is lower semi-continuous.

For χ_B, recall that since the complement of a closed set is open,

$$\chi_B = 1 - \chi_{B^c}$$

can be seen as the difference between a continuous function and a lower semi-continuous function. Since the additive inverse of a lower semi-continuous function is trivially upper semi-continuous, χ_B is upper semi-continuous. □

4. Proof of Urysohn's Lemma

Urysohn's Lemma (Paul Garrett version)**.** *Let X be a locally compact Hausdorff topological space. In X, given a compact subset K contained in an open set U, there is a continuous function $0 \leq f \leq 1$ which is 1 on K and 0 off U.*

Proof. First, choose a well-ordering $r_1, r_2, \ldots$ of the rational numbers in $[0, 1]$ with 0 and 1 as initial elements. By Urysohn's Little Lemma, we know there are V_1 and V_0 such that

$$K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset U.$$

Now, suppose we have some open sets $V_{r_1}, \ldots, V_{r_n}$ with compact closures, nested between K and U, fulfilling

$$r_p > r_q \Rightarrow V_{r_p} \subset \overline{V}_{r_p} \subset V_{r_q} \subset \overline{V}_{r_q}.$$

We'll construct $V_{r_{n+1}}$ in the following way: first, let

$$r_j = \min\{r \in \{r_1, \ldots, r_n\} : r > r_{n+1}\},$$
$$r_i = \max\{r \in \{r_1, \ldots, r_n\} : r < r_{n+1}\}.$$

Now apply Urysohn's Little Lemma, nesting $V_{r_{n+1}}$ and its closure so that the following holds:

$$V_{r_j} \subset \overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i} \subset \overline{V}_{r_i}.$$

This inductive construction has no obstruction. We end up with a collection of nested open sets of compact closure, indexed by rationals, fulfulling the following: for any $r, s \in \mathbb{Q} \cap (0,1)$,

$$r > s \Rightarrow K \subset V_r \subset \overline{V}_r \subset V_s \subset \overline{V_s} \subset U.$$

All that remains is to define our continuous function, taking advantage of these sets. Define $f : X \to [0,1]$ and $g : X \to [0,1]$ by

$$f(x) = \sup\{r \in \mathbb{Q} \cap [0,1] : x \in V_r\},$$
$$g(x) = \inf\{r \in \mathbb{Q} \cap [0,1] : x \notin \overline{V}_r\}.$$

I'll show that $f = g$ by demonstrating that

$$\{r : x \notin \overline{V}_r\} = \{r : r > f(x)\}.$$

Fix an arbitrary $x \in X$. I'll use r to denote elements of the set $\mathbb{Q} \cap [0,1]$.

By the definition of $f(x)$ (and since sets indexed by larger rationals are contained in sets indexed by smaller rationals) we know that $x \in V_r$ for all $r \leq f(x)$. Thus also $x \in \overline{V}_r$ for all $r \leq f(x)$. That is, the condition $x \notin \overline{V}_r$ is only ever possible when $r > f(x)$. So

$$\{r : x \notin \overline{V}_r\} \subseteq \{r : r > f(x)\}.$$

Also, since f is a supremum, we know that $x \notin V_r$ for all $r > f(x)$, implying also that $x \notin \overline{V}_r$ for all $r > f(x)$. This last assertion is not entirely trivial. Since the rationals are dense in [0,1] and $r > f(x)$, there exists some rational r_0 fulfilling $r > r_0 > f(x)$. Now $x \notin V_{r_0}$ since $r_0 > f(x)$, and thus $x \notin \overline{V}_r$ since $\overline{V}_r \subset V_{r_0}$. So

$$\{r : x \notin \overline{V}_r\} \supseteq \{r : r > f(x)\}.$$

So

$$\{r : x \notin \overline{V}_r\} = \{r : r > f(x)\}$$

and thus

$$g(x) = \inf\{r : x \notin \overline{V}_r\} = \inf\{r : r > f(x)\} = f(x).$$

Lastly, since f is a supremum of characteristic functions of open sets and g is an infimum of characteristic functions of closed sets, f is lower semi-continuous and g is upper semi-continuous. But $f = g$, so f is continuous. Also, $f(K) = \{1\}$ and $f(U^c) = \{0\}$ as desired. $\square$